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An efficient analytical solution to transient heat conduction in a one-dimensional hollow composite cylinder

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Abstract
In this paper a novel analytical method is applied to the problem of transient heat conduction in a one-dimensional hollow composite cylinder with a time-dependent boundary temperature. It is known that for such problems in general, the underlying eigenvalue and residue calculations pose a challenge in practice because of the computational requirements especially for a cylinder with many layers. A new approximated analytical solution is derived by a novel application of the Laplace transformation. As a result, the problem of eigenvalue or residue computation is avoided. A closed-form solution is presented. A further comparison of analytical results with numerical models demonstrates a high accuracy of the developed analytical solution.

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1. Introduction

The study of transient heat conduction in hollow composite cylinders is important in many areas of science and engineering such as thermodynamics, fuel cells, electrochemical reactors, high density microelectronics, composite materials, solidification processes and many others. Significant progress in numerical techniques to obtain solutions of heat conduction problems has solidified fundamental knowledge in many engineering applications. However, the classic way of finding analytical solutions is obviously very important in estimating material properties and validating numerical solutions.

Theoretically, analytical methods for the heat conduction problem in cylindrical and Cartesian coordinates are exactly the same. The common applied techniques are finite integral transformations which are often applied to the problem in a single-layer material: Green functions, orthogonal expansions and the Laplace transformation [1]. The first two techniques inherit associated eigenvalue problems which may become much more complicated if the slab has many layers. The third technique, the Laplace transformation, often yields a residue computation. In a composite slab, the residue computation is accomplished by directly and numerically searching for the roots of a hyperbolic equation, finding the derivatives of the equation and evaluating and summing the residues. The calculation procedure is tedious if the slab has more than two layers, as numerical searching for roots has to be made with a very fine increment for the inverse Laplace transformation to prevent missing roots which can lead to a wrong inverse [2].

Due to these mathematical complications, closed-form solutions for heat conduction problems in a composite slab are rare in literature though the studies are very extensive. In Cartesian coordinates, closed-form solutions for heat conduction equation were available for only three-layer composite slab with a constant boundary temperature in 2004 [3]. In cylindrical coordinates, Imber proposed an approximate solution in two-dimensional cylindrical geometry, which, unfortunately, is of low accuracy and therefore cannot be used in practical applications [4, 5]. Using the same technique, Monde *et al* proposed an approximate solution which can predict the surface temperature and heat flux with a good accuracy in one- and two-dimensional cases (e.g. [5]). No closed-form solution has been reported for a hollow composite cylinder with more than three layers. Moreover, in all the above-cited works, eigenvalue problems need to be solved and have always posed a challenge to analytical methods. The advantage of analytical methods over numerical methods is hard to recognize.

Recently, an analytical method was developed to tackle the transient heat problem for a composite slab in Cartesian coordinates subject to a time-dependent temperature change [6–8]. Unlike most of the traditional methods, the new analytical method involves no numerical iteration for eigenvalues and residues.

It is worth mentioning that the eigenvalue problems are often inevitable in solving transient heat conduction equations for a multi-layer slab. In a single-layer slab, the eigenfunction links the space and time variables when separation of variables is applied. However, in a multi-layer slab, eigenfunction problems may also be yielded from the boundary conditions presented in the contacted layers. Hence, eigenvalue problems may exist even for a steady-state heat conduction problem in a composite slab (e.g. [9]). A detailed literature review of these methods can be found in [8].

In this paper, the developed analytical method [6] is extended to a cylindrical geometry. The objective of this study was to derive more general closed-form solutions for the transient heat conduction problem in a hollow composite cylinder with a time-dependent temperature change. Compared to the work in cylindrical coordinates reviewed above, first, the boundary condition is given more generally. Second, there is no need to numerically search for eigenvalues and residues. Most importantly, a closed-form solution for the transient heat conduction problem in an n-layer hollow cylinder is given. A further comparison of the analytical results with the numerical models demonstrates a high accuracy of the developed analytical solution.

2. Mathematical model

2.1. Model equations

Let an n-layer composite hollow cylinder be represented by cylindrical coordinates as illustrated in figure 1. The layers are formed with different materials characterized by constant

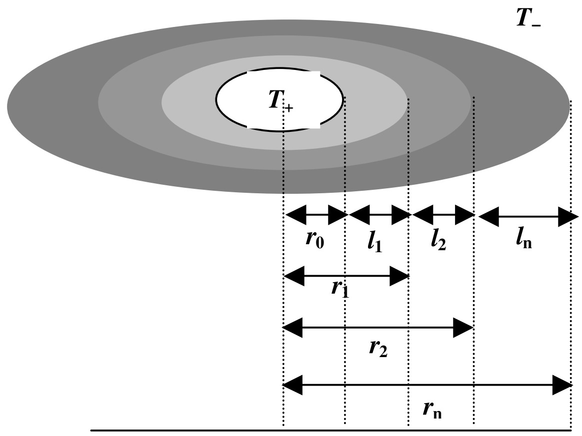

Figure 1. Schematic drawing of the composite hollow cylinder.

conductivity, diffusivity and thickness which are presented by λ_j, k_j and l_j, $j = 1, \ldots, n$. An ideal contact between these layers is assumed.

Denoting $r_j = r_0 + l_1 + \cdots + l_j$, $j = 1, \ldots, n$, the basic heat conduction equation in terms of the temperature $T_j(t, r)$ in cylindrical coordinates reads

$$\frac{\partial^2 T_j}{\partial r^2} + \frac{1}{r}\frac{\partial T_j}{\partial r} = \frac{1}{k_j}\frac{\partial T_j}{\partial t}, \qquad r \in [r_{j-1}, r_j], \qquad j = 1, \ldots, n \tag{2.1a}$$

with boundary and initial conditions

$$-\lambda_1 \frac{\partial T_1}{\partial r}(t, r_0) = -\alpha_+(T_1(t, r_0) - T_+(t)), \tag{2.1b}$$

$$T_j(t, r_j) = T_{j+1}(t, r_j), \qquad r \in [r_{j-1}, r_j], \qquad j = 1, \ldots, n-1, \tag{2.1c}$$

$$-\lambda_j \frac{\partial T_j}{\partial r}(t, r_j) = -\lambda_{j+1}\frac{\partial T_{j+1}}{\partial r}(t, r_j), \qquad r \in [r_{j-1}, r_j], \qquad j = 1, \ldots, n-1, \tag{2.1d}$$

$$-\lambda_n \frac{\partial T_n}{\partial r}(t, r_n) = -\alpha_-(T_-(t) - T_n(t, r_n)), \tag{2.1e}$$

$$T_j(0, r) = 0, \qquad r \in [r_{j-1}, r_j], \qquad j = 1, \ldots, n. \tag{2.1f}$$

Without losing generality, the initial temperature is assumed to be zero. The surface heat transfer coefficients for boundaries are denoted by α_+ and α_-. The boundary temperatures $T_+(t)$ and $T_-(t)$ are time dependent.

2.2. *Simplification of the problem*

First we assume sinusoidal dependence of the boundary temperatures, i.e. $T_+(t) = a_+ \cos(\omega_+ t + \varphi_+)$ and $T_-(t) = a_- \cos(\omega_- t + \varphi_-)$. Furthermore, a solution is given according to the complex form of the boundary temperatures, namely

$$T_+(t) = a_+ \,\mathrm{e}^{\mathrm{i}\omega_+ t + \mathrm{i}\varphi_+}, \tag{2.2a}$$

$$T_-(t) = a_- \,\mathrm{e}^{\mathrm{i}\omega_- t + \mathrm{i}\varphi_-}. \tag{2.2b}$$

Clearly, the solution of equation (2.1) is the real part of the sought-after solution. If there is no danger of confusion we shall keep the same notations. The solutions for more general boundary temperatures will be provided later.

3. Solution method

3.1. Analytical solution

Applying the Laplace transformation in equation (2.1), we get the equations in terms of $\bar{T}_j(s, r)$ as

$$\frac{\partial^2 \bar{T}_j}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{T}_j}{\partial r} = \frac{s}{k_j}\bar{T}_j, \qquad r \in [r_{j-1}, r_j], \qquad j = 1, \ldots, n, \tag{3.1a}$$

with boundary conditions

$$-\lambda_1 \frac{\partial \bar{T}_1}{\partial r}(s, r_0) = -\alpha_+(\bar{T}_1(s, r_0) - \bar{T}_+(s)), \tag{3.1b}$$

$$\bar{T}_j(s, r_j) = \bar{T}_{j+1}(s, r_j), \qquad r \in [r_{j-1}, r_j], \qquad j = 1, \ldots, n-1, \tag{3.1c}$$

$$-\lambda_j \frac{\partial \bar{T}_j}{\partial r}(s, r_j) = -\lambda_{j+1} \frac{\partial \bar{T}_{j+1}}{\partial r}(s, r_j), \qquad r \in [r_{j-1}, r_j], \qquad j = 1, \ldots, n-1, \tag{3.1d}$$

$$-\lambda_n \frac{\partial \bar{T}_n}{\partial r}(s, r_n) = -\alpha_-(\bar{T}_-(s) - \bar{T}_n(s, r_n)). \tag{3.1e}$$

A bar over a function $f(t)$ designates its Laplace transformation on t (e.g. [1]):

$$\bar{f}(s) = (f(t)) = \int_0^\infty \exp(-s\tau) f(\tau)\, d\tau. \tag{3.2a}$$

The Laplace transformation of a convolution is given by [1]

$$\mathcal{L}(f_1(t) * f_2(t)) = \bar{f}_1(s)\bar{f}_2(s) \qquad \text{where} \quad f_1(t) * f_2(t) = \int_0^t f_1(\tau) f_2(t-\tau)\, d\tau. \tag{3.2b}$$

For the *j*th layer, the solution of equation (3.1a) can be obtained by a combination of the modified Bessel functions [10] as

$$\bar{T}_j(s, r) = A_j I_0(q_j r) + B_j K_0(q_j r), \tag{3.3}$$

where $q_j = \sqrt{\frac{s}{k_j}}$, A_j and B_j are determined by the boundary conditions.

For convenience, we define the following notation:

$$h_0 = \frac{\alpha_+}{\lambda_1 q_1}, \qquad h_j = \frac{\lambda_{j+1}}{\lambda_j}\sqrt{\frac{k_j}{k_{j+1}}}, \qquad j = 1, \ldots, n-1, \qquad h_n = -\frac{\alpha_-}{\lambda_n q_n}, \tag{3.4a}$$

$$\eta_j = q_j r_{j-1}, \qquad \xi_j = q_j r_j, \qquad j = 1, \ldots, n. \tag{3.4b}$$

Inserting equation (3.3) into the boundary conditions and rearranging the resulting equations yield

$$A_1[I_1(\eta_1) - h_0 I_0(\eta_1)] - B_1[K_1(\eta_1) + h_0 K_0(\eta_1)] = -h_0 \bar{T}_+, \tag{3.5a}$$

$$A_j I_0(\xi_j) + B_j K_0(\xi_j) - A_{j+1} I_0(\eta_{j+1}) - B_{j+1} K_0(\eta_{j+1}) = 0, \qquad j = 1, \ldots, n-1, \tag{3.5b}$$

$$A_j I_1(\xi_j) - B_j K_1(\xi_j) - A_{j+1} h_j I_1(\eta_{j+1}) + B_{j+1} h_j K_1(\eta_{j+1}) = 0, \qquad j = 1, \ldots, n-1, \tag{3.5c}$$

$$A_n[I_1(\xi_n) - h_n I_0(\xi_n)] - B_n[K_1(\xi_n) + h_n K_0(\xi_n)] = -h_n \bar{T}_-. \tag{3.5d}$$

The coefficients A_j and B_j can be obtained by solving equation (3.5) as follows:

$$\Delta(s) = \begin{vmatrix} I_1(\eta_1) - h_0 I_0(\eta_1) & -K_1(\eta_1) - h_0 K_0(\eta_1) & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\ I_0(\xi_1) & K_0(\xi_1) & -I_0(\eta_2) & -K_0(\eta_2) & \ldots & 0 & 0 & 0 & 0 \\ I_1(\xi_1) & -K_1(\xi_1) & -h_1 I_1(\eta_2) & h_1 K_1(\eta_2) & \ldots & 0 & 0 & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & \ldots & I_0(\xi_{n-1}) & K_0(\xi_{n-1}) & -I_0(\eta_n) & -K_0(\eta_n) \\ 0 & 0 & 0 & 0 & \ldots & I_1(\xi_{n-1}) & -K_1(\xi_{n-1}) & -h_{n-1} I_1(\eta_n) & h_{n-1} K_1(\eta_n) \\ 0 & 0 & 0 & 0 & \ldots & 0 & 0 & I_1(\xi_n) - h_n I_0(\xi_n) & -K_1(\xi_n) - h_n K_0(\xi_n) \end{vmatrix}, \tag{3.6a}$$

$$\Delta_j^1(s) = -h_0 \frac{\begin{vmatrix} \Delta(s) & \text{with} \\ \text{row} - 1 & \text{column} - 2j - 1 \\ \text{deleted} & \end{vmatrix}}{\Delta(s)}, \qquad \Delta_j^2(s) = h_n \frac{\begin{vmatrix} \Delta(s) & \text{with} \\ \text{row} - 2n & \text{column} - 2j - 1 \\ \text{deleted} & \end{vmatrix}}{\Delta(s)}, \tag{3.6b}$$

$$\Delta_j^3(s) = h_0 \frac{\begin{vmatrix} \Delta(s) & \text{with} \\ \text{row} - 1 & \text{column} - 2j \\ \text{deleted} & \end{vmatrix}}{\Delta(s)}, \qquad \Delta_j^4(s) = -h_n \frac{\begin{vmatrix} \Delta(s) & \text{with} \\ \text{row} - 2n & \text{column} - 2j \\ \text{deleted} & \end{vmatrix}}{\Delta(s)}. \tag{3.6c}$$

$$A_j = \Delta_j^1 \bar{T}_+ + \Delta_j^2 \bar{T}_-, \qquad B_j = \Delta_j^3 \bar{T}_+ + \Delta_j^4 \bar{T}_-. \tag{3.7}$$

Equation (3.3) is rewritten as

$$\begin{aligned} \bar{T}_j(s, r) &= A_j I_0(q_j r) + B_j K_0(q_j r) \\ &= \left[\Delta_j^1 I_0(q_j r) + \Delta_j^3 K_0(q_j r)\right]\bar{T}_+ + \left[\Delta_j^2 I_0(q_j r) + \Delta_j^4 K_0(q_j r)\right]\bar{T}_- \\ &= F_j(s, r)\bar{T}_+ + G_j(s, r)\bar{T}_-, \end{aligned} \tag{3.8a}$$

where

$$F_j(s, r) = \Delta_j^1 I_0(q_j r) + \Delta_j^3 K_0(q_j r), \qquad G_j(s, r) = \Delta_j^2 I_0(q_j r) + \Delta_j^4 K_0(q_j r). \tag{3.8b}$$

The next step is to get the inverse of $\bar{T}_j(s,r)$. Let $f_j(t,r)$ and $g_j(t,r)$ be the inverse Laplace transformations of $F_j(s,r)$ and $G_j(s,r)$; then equation (3.8*a*), together with the property of the Laplace transformation (3.2*b*), gives

$$
\begin{aligned}
T_j(t,r) &= f_j * T_+ + g_j * T_- = \int_0^t (f_j(\tau,r)T_+(t-\tau) + g(\tau,r)_j T_-(t-\tau))\,\mathrm{d}\tau \\
&= \int_0^\infty - \int_t^\infty (f_j(\tau,r)T_+(t-\tau) + g_j(\tau,r)T_-(t-\tau))\,\mathrm{d}\tau \\
&\approx \int_0^\infty (f_j(\tau,r)T_+(t-\tau) + g_j(\tau,r)T_-(t-\tau))\,\mathrm{d}\tau \\
&= \int_0^\infty \{f_j(\tau,r)a_+ \exp[\mathrm{i}(\omega_+(t-\tau)+\varphi_+)] + g_j(\tau,r)a_- \exp[\mathrm{i}(\omega_- - (t-\tau)+\varphi_-)]\}\,\mathrm{d}\tau \\
&= a_+ \exp[\mathrm{i}(\omega_+ t + \varphi_+)] \int_0^\infty \underbrace{\exp(-\mathrm{i}\omega_+\tau) f_j(\tau,x)\,\mathrm{d}\tau}_{\uparrow \atop \text{Laplace transformation of } f \text{ at } \mathrm{i}\omega_+} \\
&\quad + a_- \exp[\mathrm{i}(\omega_- t + \varphi_-)] \int_0^\infty \underbrace{\exp(-\mathrm{i}\omega_-\tau) g_j(\tau,x)\,\mathrm{d}\tau}_{\uparrow \atop G_j(\mathrm{i}\omega_-, r)} \\
&= F_j(\mathrm{i}\omega_+, r)T_+ + G_j(\mathrm{i}\omega_-, r)T_-.
\end{aligned}
\tag{3.9}
$$

An approximated analytical solution $T_j(t,r)$ has been obtained. Note that a mathematical trick is made in the above calculation. Taking advantage of the mathematical expression of the exponential functions T_+ and T_-, $f(t,r)$ and $g(t,r)$ are replaced by their Laplace transformations which are already available. Hence, $f_j(t,r)$ and $g_j(t,r)$ are acting only as symbolic functions. In this way, a complicated residue calculation is avoided.

Note that the boundary temperatures are in their complex forms. So the final solution for the *j*th layer is

$$
T_j(t,r) = \mathrm{real}(F_j(\mathrm{i}\omega_+, r)T_+ + G_j(\mathrm{i}\omega_-, r)T_-), \qquad r \in [r_{j-1}, r_j], \qquad j = 1, \ldots, n. \tag{3.10}
$$

where ‘real’ represents the real part of the function.

3.2. Solution for more general boundary temperature

For more general time-dependent boundary temperatures, we approximate the temperatures by Fourier series, i.e.

$$
T_+(t) = a_{+0} + \sum_{k=1}^{\infty} a_{+k} \cos(\omega_{+k} t + \varphi_{+k}), \tag{3.11a}
$$

$$
T_-(t) = a_{-0} + \sum_{k=1}^{\infty} a_{-k} \cos(\omega_{-k} t + \varphi_{-k}). \tag{3.11b}
$$

The solution of the system equation is the sum of the solution with the constant boundary temperatures a_{+0} and a_{-0} and the solution with boundary temperatures $\sum_{k=1}^{\infty} a_{+k} \cos(\omega_{+k} t + \varphi_{+k})$ and $\sum_{k=1}^{\infty} a_{-k} \cos(\omega_{-k} t + \varphi_{-k})$. The second part of the solution is easily obtained according to the above-discussed theory due to linearity of the equation system.

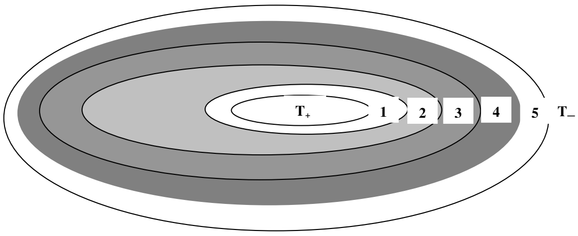

Figure 2. Schematic drawing of the five-layer hollow cylinder.

For the first part of the solution with constant boundary temperatures $T_+ = a_{+0}$ and $T_- = a_{-0}$, the calculation procedures for the solution are analogous. Hence, equation (3.8) reads

$$\begin{aligned}\bar{T}_j(s,r) &= F_j(s,r)\bar{T}_+ + G_j(s,r)\bar{T}_- + \cdots \\ &= F_j(s,r)\frac{a_{+0}}{s} + G_j(s,r)\frac{a_{-0}}{s} + \cdots .\end{aligned} \tag{3.12}$$

The inverse of the omitted term presents the second solution. Determinants F_j and G_j are the functions of hyperbolic functions sine and cosine which can be approximated by power series. Linearization of equation (3.12) gives

$$\bar{T}_j(s,r) \approx \frac{\mathrm{const}}{\mathrm{const}\,1 * s + \mathrm{const}\,2} + \cdots . \tag{3.13}$$

The inverse of the first part of the solution is then obtained as $\frac{\mathrm{const}}{\mathrm{const}\,1}\exp\left(-\frac{\mathrm{const}\,2}{\mathrm{const}\,1}t\right)$.

A simpler way of finding the first part of the solution with constant boundary temperatures is to ignore the transient term which will die away if studies do not focus on the initial temperature change. Then the solution is approximated by the steady-state one which can be easily obtained from the thermal resistance of the n-layer hollow cylinder [1].

4. Example

4.1. Material parameters

A five-layer hollow composite cylinder was selected to demonstrate the calculations. The schematic picture is shown in figure 2. The composition is an extension of the three-layer slab employed as an exterior wall structure in our test house which has been used to evaluate the accuracy of the developed analytical method for predicting the evolution of the temperature distributions inside the wall subject to interior and exterior climatic conditions. Normally, the interior temperature is assumed to be constant and the exterior temperatures are measured. The main material of the wall is mineral wool (200 mm) with the wall paper (25 mm) and the gypsum board (13 mm) as boundary materials. These materials are presented as layers 1, 4 and 5 in figure 2 and their physical properties are provided in table 1. More practical background of this example has been given in [9].

In the calculation example presented in this paper, the three-layer wall construction is extended to three-layer and five-layer structures in cylindrical coordinates. In the five-layer composition displayed in figure 2, two more materials (brick and concrete) are added whose physical properties are provided in table 1. We compare only the analytical and numerical results for the five-layer composition, because (1) the comparison results for the three-layer composition have not shown any substantial change and (2) the five-layer composition can better demonstrate the capability of the developed analytical method.

Table 1. Properties and dimensions of the composite hollow cylinder.

Layer	Thermal conductivity ($W\ m^{-1}\ K^{-1}$)	Thermal diffusivity ($m^2\ s^{-1}$)	Thickness (mm)
1	0.23	4.11×10^{-7}	50
2	0.0337	1.47×10^{-6}	100
3	0.9	3.75×10^{-7}	100
4	0.147	1.61×10^{-7}	200
5	0.12	1.5×10^{-7}	20

Table 2. Parameters of equation (4.1).

	ω_1 30.0	ω_2 5.0	ω_3 2.0	ω_4 1.0
	φ_{+1} 5.149 231	φ_{+2} 16.779 94	φ_{+3} −0.678 84	φ_{+4} 4.381 328
a_{+0} 17.0	a_{+1} 1.919 486	a_{+2} 0.732 953	a_{+3} −0.258 24	a_{+4} 0.132 831
	φ_{-1} 5.607 506	φ_{-2} 13.595 96	φ_{-3} 1.451 539	φ_{-4} 5.418 717
a_{-0} 5.0	a_{-1} 2.722 17	a_{-2} −5.019 664	a_{-3} 1.084 058	a_{-4} 0.4648

The dimensions of the composition are given in table 1. The surface heat transfer coefficients were assumed to be $\alpha_- = 25\ W\ m^{-2}\ K^{-1}$ and $\alpha_+ = 6\ W\ m^{-2}\ K^{-1}$.

4.2. Boundary temperatures

As stated earlier, we are interested in predicting the evolution of the temperature distributions inside the structure subject to interior and exterior climatic conditions. Hence, in the example, the boundary temperatures were taken from the measurements and then fitted with periodic functions with periods 30, 5, 2 and 1 day as

$$T_+(t) = a_{+0} + \sum_1^4 a_{+i} \cos\left(\frac{2\pi t}{\omega_i} - \varphi_{+i}\right), \qquad (4.1a)$$

$$T_-(t) = a_{-0} + \sum_1^4 a_{-i} \cos\left(\frac{2\pi t}{\omega_i} - \varphi_{-i}\right), \qquad (4.1b)$$

where fitting parameters are listed in table 2 and figure 3 shows the values.

The time period of the measurement was 30 days. The fitting periods, 30, 5, 2 and 1 day, were just randomly chosen and the fitting parameters were generated then by using Microsoft Excel software. Theoretically, any piecewise function can be approximated by Fourier series. To the authors' knowledge, there are many kinds of software which are able to approximate a common function using Fourier series by, for instance, the fast Fourier transform (FFT) method. The more Fourier terms that are included the better the approximation. Functions with sharp corners need more Fourier terms. Therefore, the needed number of terms depends very much on the desired accuracy. Because the main purpose of this study was to evaluate the accuracy of the developed analytical solution, a rough approximation of the boundary condition was made in this calculation. Discussion of the methods of approximating an arbitrary function by Fourier series is outside the scope of this study. Interested readers can refer to the relevant textbooks (e.g. [11]).

4.3. Comparison of the analytical and numerical results

The calculations were made in the central points of layers represented as layer 1 to layer 5 in figure 2. As the evolution of temperature in layer 5 is very close to that in layer 4, only

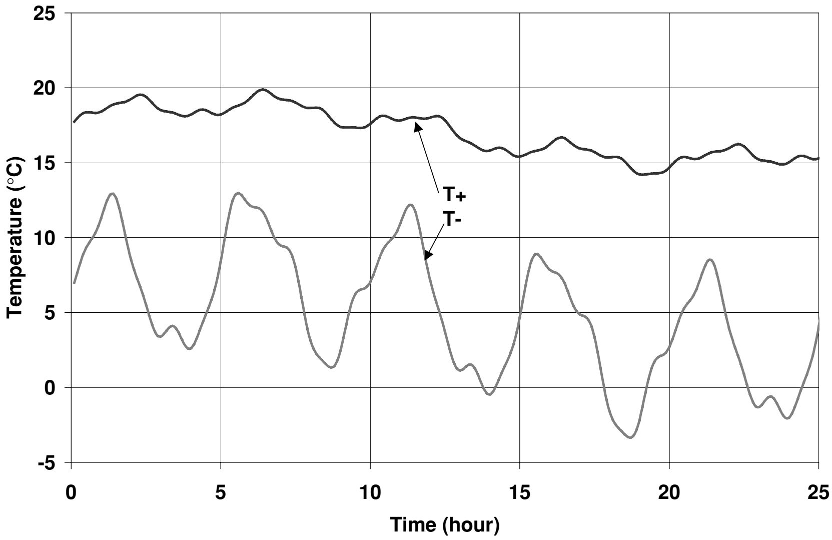

Figure 3. Variation of boundary temperature.

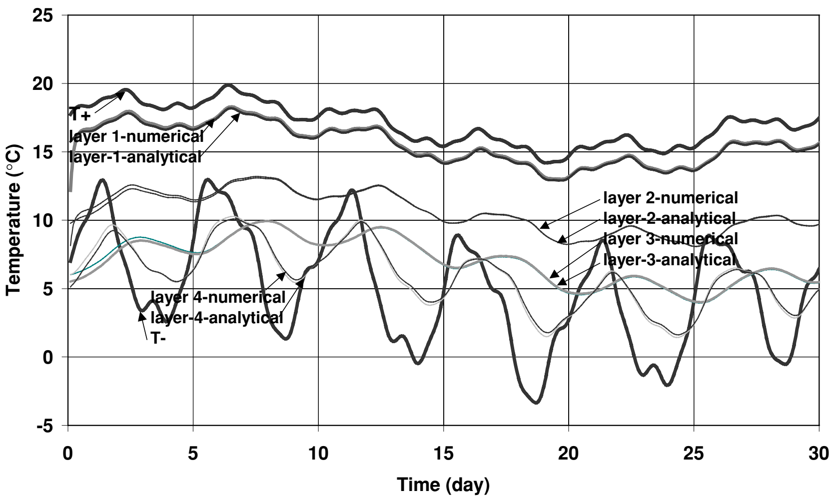

Figure 4. Comparison of analytical and numerical results.

the transient temperature variations in layer 1 to layer 4 are illustrated. Figure 4 presents the results given by the analytical and numerical methods. The temperature values were calculated so that the time step was in seconds and were stored in files as hourly values and shown in figures as hourly and daily values. It can be seen that the discrepancies between numerical and analytical results are hardly noticeable. The maximal discrepancy is within 1.5 °C at the time just after $t = 0$ with a relative error of 6%. This is mainly due to the rough estimation of the initial temperature distribution in the composition in the numerical calculations.

The bigger discrepancies of the analytical and numerical results are shown in detail in figure 5. The numerical values quickly approached the analytical values. The example

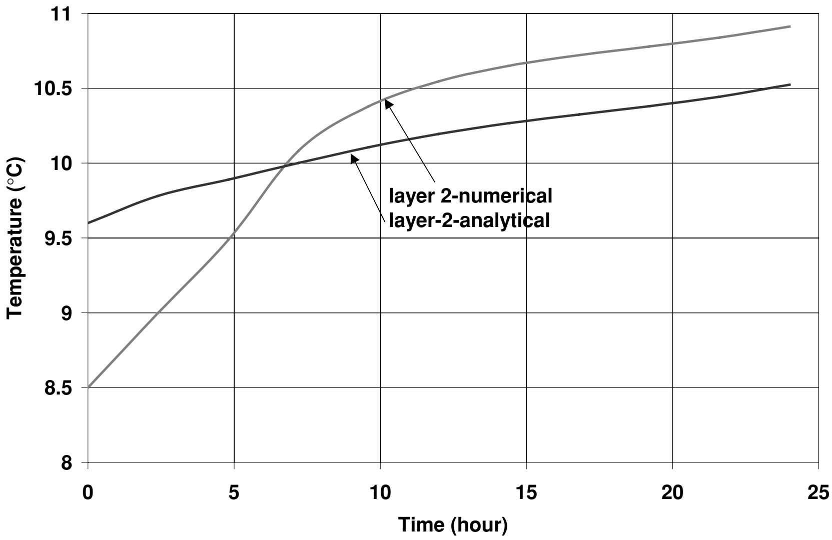

Figure 5. Comparison of analytical and numerical results, details of figure 4.

demonstrates a high accuracy of the developed analytical solution. Discussion of the validation of the numerical model can be found in [6] for example.

5. Discussion

A closed-form solution of transient heat conduction in an n-layer hollow cylinder is provided. We make some observations.

- It is known that any periodic and piecewise continuous function can be approximated using Fourier series. A non-periodic function can also be approximated using Fourier series in an extended interval. As demonstrated in the example, the boundary temperatures can be any time-dependent functions. In the example, the boundary temperatures were taken from the measurements and then fitted with a sum of cosines. Therefore, the analytical results obtained in this paper are the solutions for heat conduction problems in a composite cylinder with general boundary conditions. Boundary conditions are not restricted to periodic ones as demonstrated in the example.
- Compared with numerical methods, the developed analytical method is easier to implement. The calculation includes only simple computation of determinants which can be easily accomplished by commercial mathematical packages such as Maple, Matlab and Mathematica. Furthermore, for any jth layer, only five sparse matrices are involved. The calculation load is small and the computing time is short.
- For any jth layer and a given boundary temperature, $F_j(s,r)$ and $G_j(s,r)$ in equation (3.10) are acting as 'transfer' functions from which the attenuated temperature amplitude and the time lag in the jth layer can be obtained. Moreover, expressing $F_j(s,r)$ and $G_j(s,r)$ as the algebraic functions of parameter k_j, for instance, the effect of different physical properties on the temperature can be studied and analysed.
- The proposed method relies on the approximations in equations (3.9) and (3.13). For small values of t, the accuracy may be low. But we have not found any calculational evidence. Comparison of analytical and numerical results shows that the analytical solutions are

accurate also for small values of t. The estimation of the magnitudes of the omitted terms in equations (3.9) and (3.13) will be done in the future.

- In engineering applications, the accuracy of the developed method depends on that of the Fourier series approximation of the boundary temperatures. Better approximation of the temperatures can improve the accuracy of the solution. Nevertheless, it is worth mentioning that the accuracy problem is not an inherent error due to the analytical method developed in this paper.

6. Conclusions

An analytical approach to the heat conduction problem in a composite hollow cylinder with a general time-dependent boundary condition has been presented. The boundary temperatures were approximated using Fourier series. The technique of Laplace transform was employed. An approximate analytical solution was obtained by approximating the inverse Laplace transform without evaluating residues.

The method is shown to have considerable potential in solving heat conduction equations. Compared to the analytical solutions that are mainly possible with certain assumptions, the developed method has no restrictions and is flexible with regard to boundary conditions. Moreover, the benefit of the results is the simple and concise mathematical forms of the solutions which can be used to analyse physical properties in combination with material properties in heat transfer process. Agreement with numerical solutions is good. In a general heat conduction context, however, numerical schemes are usually necessary. The proposed approach is free of these restrictions.

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